

Large Deviations Principle for a Large Class of One-Dimensional Markov Processes

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Abstract

We study the large deviations principle for one dimensional, continuous, homogeneous, strong Markov processes that do not necessarily behave locally as a Wiener process. Any strong Markov process X_t in \mathbb{R} that is continuous with probability one, under some minimal regularity conditions, is governed by a generalized elliptic operator $D_v D_u$, where v and u are two strictly increasing functions, v is right continuous and u is continuous. In this paper, we study large deviations principle for Markov processes whose infinitesimal generator is $\epsilon D_v D_u$ where $0 < \epsilon \ll 1$. This result generalizes the classical large deviations results for a large class of one dimensional "classical" stochastic processes. Moreover, we consider reaction-diffusion equations governed by a generalized operator $D_v D_u$. We apply our results to the problem of wave front propagation for these type of reaction-diffusion equations.

Key words: Large deviations principle, Action functional, Strong Markov processes in one dimension, Wave front propagation, Reaction - diffusion equations.

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1 Introduction

It is well known that for each classical second order differential operator

$$Lf(x) = \frac{1}{2}a(x)\frac{d^2 f(x)}{dx^2} + b(x)\frac{df(x)}{dx} \quad (1.1)$$

with smooth enough coefficients $a(x) > 0$ and $b(x)$, there exists a diffusion process (X_t, \mathbb{P}_x) in \mathbb{R} such that L is the generator of this process. The domain of definition of L is $\mathcal{D}(L) = \{f : f \in \mathcal{C}^2(\mathbb{R})\}$. If $a(x), b(x) \in \mathcal{C}(\mathbb{R})$ with $a(x) > 0$, the trajectories of X_t can be constructed as the solutions of the following stochastic differential equation:

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x, \quad (1.2)$$

where $a(x) = \sigma^2(x)$ and W_t is the standard Wiener process in \mathbb{R} . It is also widely known that if X_t satisfies (1.2) then it behaves locally like a Wiener process. In particular, it spends zero time at any given point $x \in \mathbb{R}$ and it exits the interval $[x - \delta, x + \delta]$ through both ends with asymptotically equal probabilities as $\delta \downarrow 0$.

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Let now $0 < \epsilon \ll 1$ be a small positive number. Denote by X_t^ϵ the process that is governed by the operator

$$L^\epsilon f(x) = \frac{\epsilon}{2} a(x) \frac{d^2 f(x)}{dx^2} + b(x) \frac{df(x)}{dx}. \quad (1.3)$$

Then, large deviations principle for the process X_t^ϵ is well known (Freidlin and Wentzel [8]; see also [5] and [11]). In particular, the action functional for the process $(X_t^\epsilon)_{t \in [0, T]}$, in $\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$ has the form $\frac{1}{\epsilon} S_{0T}(\phi)$, where

$$S_{0T}(\phi) = \begin{cases} \frac{1}{2} \int_0^T \frac{|\dot{\phi}_s - b(\phi_s)|^2}{a(\phi_s)} ds, & \text{if } \phi \in \mathcal{C}([0, T]; \mathbb{R}) \text{ is absolutely continuous} \\ +\infty, & \text{for the rest of } \mathcal{C}([0, T]; \mathbb{R}). \end{cases} \quad (1.4)$$

However, no general results on large deviations principle are known for general one-dimensional, strong Markov processes that do not behave locally as a Wiener process. Namely, for processes that may spend positive time at a given point $x \in \mathbb{R}$ or that may exit a given interval $[x - \delta, x + \delta]$ with unequal probabilities from left and right as $\delta \downarrow 0$. The purpose of this paper is to study exactly this situation for a large class of one dimensional, homogeneous, strong Markov processes that are continuous with probability one. These processes were characterized by Feller [3] in a unique way through a generalized second order elliptic operator $D_v D_u$ and its domain of definition.

As we shall also see below, the functions v and u that appear in the $D_v D_u$ operator are in general non smooth. Function u could be non differentiable and function v could even have jump discontinuities. Note that if they were sufficiently smooth, then one would recover the classical second order operator (1.3) (see below for more details). These non-smoothness issues create several technical difficulties in the proof of the large deviations principle that one has to overcome. We overcome these difficulties and we provide an explicit expression for the action functional which is in terms of the u and the v functions under minimal assumptions on u and v . Moreover, we apply our results to the problem of wave front propagation for reaction diffusion equations where the operator of the partial differential equation is a generalized elliptic operator $D_v D_u$. Such reaction diffusion equations can appear in applications as, for example, the limit of a family of standard reaction-diffusion equations where the diffusion and drift coefficients converge to non-smooth functions. Then, as we shall also see in section 4, the characterization of the limit through a $D_v D_u$ operator is very convenient and one can use the expression for the action functional to calculate the position of the wave front. Moreover, the non-smoothness of the v and u functions can create several phenomena in the propagation of the front such as change in the speed of the propagation.

In addition, such $D_v D_u$ processes arise naturally in applications as limits of diffusion processes. For example, we mention: (a) the limiting process for nondegenerate diffusion in narrow branching tubes with reflection at the boundary (see Freidlin and Wentzel [9]) and (b) the Wiener process with reflection in non-smooth narrow tubes (see Spiliopoulos [18]). In both cases, the diffusion process in the narrow branching tube or in the narrow non-smooth tube (for (a) and (b) respectively) converges weakly to a strong Markov process X_t , as the tube becomes thinner and thinner. The limiting process behaves like a standard diffusion process on the left and on the right of the point where the branching occurs or of the discontinuity point (for (a) and (b) respectively) and has to satisfy a gluing condition at that point. Knowing the action functional for these kind of processes, one can study several other problems of interest. We mention, for example: (i) exit problems, (ii) wave front propagation for reaction diffusion equations where the operator of the partial differential equation is a generalized elliptic operator $D_v D_u$ and other related problems.

In this paper we study the large deviations principle for a one dimensional strong Markov process X_t^ϵ with generator $\epsilon D_v D_u$, where $u(x)$ and $v(x)$ are given functions, and $X_0^\epsilon = x$. In particular, $u(x)$ and $v(x)$ are strictly increasing functions, $u(x)$ is continuous and $v(x)$ is right continuous and D_v, D_u are differentiation operators with respect to v and u respectively. The expression for the action functional is in Theorem 1.1. Corollary 1.2 gives an equivalent and simpler expression for the action functional under some stricter assumptions. These results generalize the classical large deviations results for a large class of one dimensional strong Markov processes that cannot be expressed as solutions to stochastic differential equations. In particular, Corollary 1.5 shows that our form of the action functional reduces to (1.4) with $b = 0$, if u and v have a special form and enough smoothness is provided.

Before mentioning the main result of this paper (Theorem 1.1) we need to introduce some notation. Let us define the sets

$$\begin{aligned} U &= \{x \in \mathbb{R} : \text{the derivative of } u \text{ does not exist at } x\} \\ V &= \{x \in \mathbb{R} : \text{the derivative of } v \text{ does not exist at } x, v \text{ is continuous at } x\} \\ V_d &= \{x \in \mathbb{R} : v \text{ is discontinuous at } x\} \end{aligned} \quad (1.5)$$

Of course, the sets U, V and V_d are at most countably infinite.

Moreover, for a continuous function $\phi : [0, T] \rightarrow \mathbb{R}$, i.e. $\phi \in \mathcal{C}([0, T]; \mathbb{R})$, we define the sets

$$\begin{aligned} U_\phi &= \{t \in [0, T] : \phi(t) \in U\} \\ V_\phi &= \{t \in [0, T] : \phi(t) \in V\} \\ V_{d,\phi} &= \{t \in [0, T] : \phi(t) \in V_d\}. \end{aligned} \quad (1.6)$$

We also define the sets

$$E = (U \cup V) \setminus V_d \quad \text{and} \quad E_\phi = (U_\phi \cup V_\phi) \setminus V_{d,\phi}. \quad (1.7)$$

Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $u(x)$ and $v(x)$ be strictly increasing functions, $u(x)$ be continuous and $v(x)$ be right continuous. Assume that there are positive constants c_1 and c_2 such that $0 < u'(x) \leq c_1$ and $0 < c_2 \leq v'(x)$ at the points x where the derivatives of $u(x)$ and $v(x)$ exist. Let X_t^ϵ be the strong Markov process whose infinitesimal generator is $\epsilon D_v D_u$ for $0 < \epsilon \ll 1$ with initial point $X_0^\epsilon = x$.*

Let $\phi : [0, T] \rightarrow \mathbb{R}$ be a continuous function in $[0, T]$. We have the following.

(i). *If the Lebesgue measure of the set E_ϕ is zero, i.e. $\Lambda(E_\phi) = 0$, then*

$$\sigma_\phi(t) = \int_0^t \left[\frac{1}{2} \frac{dv}{du}(\phi_s) \right]^{-1} ds \quad (1.8)$$

is well defined, it is continuous and non-decreasing in t . If $\Lambda(V_{d,\phi}) = 0$, then $\sigma_\phi(t)$ is strictly increasing in t . For functions ϕ such that $\Lambda(E_\phi) > 0$ we interpret, without loss of generality, the derivative $\frac{dv}{du}$ in the formula for $\sigma_\phi(t)$ as the minimum of the left and right derivatives of v with respect to u on the countable set E (see Remark 1.3 and the statement of Lemma 2.3 for more details).

(ii). Denote by $\gamma_\phi(t)$ the generalized inverse to $\sigma_\phi(t)$, i.e.

$$\gamma_\phi(t) = \inf\{s : \sigma_\phi(s) > t\}. \quad (1.9)$$

The action functional for the process $(X_t^\epsilon)_{t \in [0, T]}$, in $\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$ has the form $\frac{1}{\epsilon} S_{0T}(\phi)$ where

$$S_{0T}(\phi) = \begin{cases} \frac{1}{2} \int_0^{\sigma_\phi(T)} \left| \frac{du(\phi(\gamma_\phi(s)))}{ds} \right|^2 ds, & \text{if } u(\phi(\gamma_\phi(s))) \text{ is absolutely continuous and } \phi_0 = x \\ +\infty, & \text{for the rest of } \mathcal{C}([0, T]; \mathbb{R}). \end{cases} \quad (1.10)$$

The functional $S_{0T}(\phi)$ is lower semi-continuous in the sense of uniform convergence. Namely, if a sequence ϕ^n converges uniformly to ϕ in $\mathcal{C}([0, T]; \mathbb{R})$, then $S_{0T}(\phi) \leq \liminf_{n \rightarrow \infty} S_{0T}(\phi^n)$. Lastly, the set $\Phi_s = \{\phi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}(\phi) \leq s \text{ and } \phi(0) \text{ belongs to a compact subset of } \mathbb{R}\}$ is compact.

□

The following corollary gives a useful representation of the action functional in the case where v is a continuous function. Then, of course, $V_d = \emptyset$, $E = U \cup V$ and $\sigma_\phi(t)$ is strictly increasing. It follows directly from Theorem 1.1 after a straightforward change of variables.

Corollary 1.2. *In addition to the assumptions of Theorem 1.1, let us assume that the function $v(x)$ is continuous. The action functional for the process $(X_t^\epsilon)_{t \in [0, T]}$, in $\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$ is $\frac{1}{\epsilon} S_{0T}(\phi)$ where*

$$S_{0T}(\phi) = \begin{cases} \frac{1}{4} \int_0^T (u \circ \phi)'(s)(v \circ \phi)'(s) ds, & \text{if } \phi \text{ is absolutely continuous and } \phi_0 = x \\ +\infty, & \text{for the rest of } \mathcal{C}([0, T]; \mathbb{R}). \end{cases} \quad (1.11)$$

Moreover, note that for ϕ absolutely continuous we have $S_{E_\phi}(\phi) = 0$.

□

Remark 1.3. As we saw in the statement of Theorem 1.1 part (i), $\sigma_\phi(t)$ is well defined for ϕ such that $\Lambda(E_\phi) = 0$. As a consequence, the action functional is also well defined. For ϕ such that $\Lambda(E_\phi) > 0$ we defined $\sigma_\phi(t)$ using formula (1.8) by interpreting the derivative $\frac{dv}{du}$ as the minimum of the left and right derivatives of v with respect to u on the countable set E . This is done without loss of generality. In particular, let us pick a point $z \in (U \cup V) \setminus V_d$ and denote $E_\phi^z = \{t \in [0, T] : \phi_t = z\}$. Then, for ϕ absolutely continuous, we have $S_{E_\phi^z}(\phi) = 0$ (independently of the interpretation of the u and v derivatives on E). More details will be given in the proof of Theorem 2.10.

□

For the convenience of the reader, we briefly recall the Feller characterization of all one-dimensional Markov processes, that are continuous with probability one (for more details see [3]; also [15]). All one-dimensional strong Markov processes that are continuous with probability one, can be characterized (under some minimal regularity conditions) by a generalized second order differential operator $D_v D_u f$ with respect to two increasing functions $u(x)$ and $v(x)$; $u(x)$ is continuous, $v(x)$ is right continuous. In addition, D_u , D_v are differentiation operators with respect to $u(x)$ and $v(x)$ respectively, which are defined as follows:

$D_u f(x)$ exists if $D_u^+ f(x) = D_u^- f(x)$, where the left derivative of f with respect to u is defined as follows:

$$D_u^- f(x) = \lim_{h \downarrow 0} \frac{f(x-h) - f(x)}{u(x-h) - u(x)} \quad \text{provided the limit exists.}$$

The right derivative $D_u^+ f(x)$ is defined similarly. If v is discontinuous at y then

$$D_v f(y) = \lim_{h \downarrow 0} \frac{f(y+h) - f(y-h)}{v(y+h) - v(y-h)}.$$

Remark 1.4. For example, it is easy to see that the operator L in (1.1) can be written as a $D_v D_u$ operator with u and v as follows:

$$u(x) = \int_0^x e^{-\int_0^y \frac{2b(z)}{a(z)} dz} dy \quad \text{and} \quad v(x) = \int_0^x \frac{2}{a(y)} e^{\int_0^y \frac{2b(z)}{a(z)} dz} dy. \quad (1.12)$$

The representation of $u(x)$ and $v(x)$ in (1.12) is unique up to multiplicative and additive constants. In fact, one can multiply one of these functions by some constant and divide the other function by the same constant or add a constant to either of them.

□

Corollary 1.5 is easily obtained from Corollary 1.2 and Remark 1.4. It shows in which way the action functional in (1.4) is generalized by the functional in (1.10) in the case of $b = 0$.

Corollary 1.5. If $u(x)$ and $v(x)$ are given by (1.12) and $a(x)$, $b(x)$ are regular enough, then $E_\phi = \emptyset$ and the action functional in (1.10), or equivalently in (1.11), coincides with (1.4) with $b = 0$.

□

The rest of the paper is organized as follows. In section 2, we prove that (1.10) is the action functional for $(X_t^\epsilon)_{t \in [0, T]}$ assuming that (1.8) is well defined. In section 3, we prove: (a) that $\sigma_\phi(t)$ in (1.8) is well defined for functions ϕ such that the Lebesgue measure of the set E_ϕ is zero and (b) several auxiliary results that are used in section 2 to prove Theorem 1.1. In section 4, we consider reaction-diffusion equations governed by a generalized operator $D_v D_u$ and we apply our results to the problem of wave front propagation for these type of reaction-diffusion equations. Lastly, section 5 includes some concluding comments and remarks on future work.

2 Estimates for probabilities of large deviations

In this section we prove that (1.10) is the action functional for $(X_t^\epsilon)_{t \in [0, T]}$. However, first we introduce some notation that we will use throughout the paper and we state the results of [19] that

we use. Then we state without proof some auxiliary results. The proof of these auxiliary lemmas will be given in the next section.

In this and the following sections we will denote by C_0 any unimportant constants that do not depend on any small or big parameter. The constants may change from place to place though, but they will always be denoted by the same C_0 . Moreover, we fix two functions $u(x)$ and $v(x)$ that have the properties of Theorem 1.1 and we denote by X_t^ϵ for the process whose infinitesimal generator is $\epsilon DvDu$. Additionally, let $u_{-1}(x)$ denoting the inverse function of $u(x)$.

Furthermore, for a continuous function $\phi : [0, T] \rightarrow \mathbb{R}$ we define the functions $\sigma_{u_{-1}(\phi)}(t)$ and $\gamma_{u_{-1}(\phi)}(t)$ in the same way to (1.8) and (1.9) with $u_{-1}(\phi)$ in place of ϕ .

The following key result is a restatement of Theorem 4 in [19].

Theorem 2.1. *Let $u(x)$ and $v(x)$ be strictly increasing functions, $u(x)$ be continuous and $v(x)$ be right continuous. Let $(v_n(x))_{n \in \mathbb{N}}$ be a sequence of strictly increasing functions, continuously differentiable with respect to $u(x)$ and converging to $v(x)$ at every continuity point of $v(x)$. Moreover, W_t denotes the standard one dimensional Wiener process.*

We introduce the variables $\tau_{u_{-1}(W)}^n(t)$ by the equations

$$\int_0^{\tau_{u_{-1}(W)}^n(t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(W_s)) ds = t \quad (2.1)$$

Then we have:

(i). *$\lim_{n \rightarrow \infty} \tau_{u_{-1}(W)}^n(t)$ exists uniformly in $t \geq 0$ on any finite time interval in the sense of convergence in probability, for all measures \mathbb{P}_x and independently of the choice of the sequence $(v_n)_{n \in \mathbb{N}}$. Moreover, $\lim_{n \rightarrow \infty} \tau_{u_{-1}(W)}^n(t)$ is strictly increasing in t with \mathbb{P}_x probability 1.*

(ii). *Denote*

$$\tau_{u_{-1}(W)}(t) = \lim_{n \rightarrow \infty} \tau_{u_{-1}(W)}^n(t) \quad (2.2)$$

The process

$$X_t = u_{-1}[W_{\tau_{u_{-1}(W)}(t)}] \quad (2.3)$$

is a homogeneous, strong Markov process whose infinitesimal generator is $D_v D_u$. The domain of definition of the $D_v D_u$ operator is

$$\begin{aligned} \mathcal{D}(D_v D_u) &= \{f : f \in \mathcal{C}_c(\mathbb{R}), \text{ where at each non smoothness point} \\ &\quad x_i \text{ of } u \text{ and } v \text{ the gluing condition holds} \\ &\quad D_u^+ f(x_i) - D_u^- f(x_i) = [v(x_i+) - v(x_i-)] D_v D_u f(x_i) \\ &\quad \text{and } D_v D_u f(x_i) = \lim_{x \rightarrow x_i^+} D_v D_u f(x) = \lim_{x \rightarrow x_i^-} D_v D_u f(x)\}. \end{aligned} \quad (2.4)$$

□

Remark 2.2. *Theorem 2.1 essentially says that any continuous, homogeneous, strong Markov process that can be characterized through a $D_v D_u$ operator, can be obtained from a Wiener process after a random time change and a space transformation. Moreover, a simple application of Itô formula shows that if $u(x)$ and $v(x)$ are given by (1.12) and $a(x), b(x)$ are regular enough, then $X_t = u_{-1}[W_{\tau_{u_{-1}(W)}(t)}]$ satisfies (1.2).*

□

We will also need the following results whose proof will be given in the next section. Lemma 2.3 is essentially part (i) of Theorem 1.1. Lemmas 2.4 and 2.5 are technical lemmas that will be used in the proof of lower semicontinuity of the functional $S_{0T}(\phi)$ and compactness of the set $\Phi_s = \{\phi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}(\phi) \leq s\}$. Proposition 2.6 gives a representation of the process X_t^ϵ that is governed by the generator $\epsilon D_v D_u$ in the spirit of Theorem 2.1. Lemma 2.7 discusses the exponential tightness of $Y_t^\epsilon = u(X_t^\epsilon)$. Using the aforementioned results we prove Theorems 2.9 and 2.10 which discuss the large deviations principle for $Y_t^\epsilon = u(X_t^\epsilon)$.

The proof of Theorem 1.1 follows from Remark 1.3, Theorems 2.9 and 2.10 and the well known contraction principle for large deviations. Namely, we find the action functional of X_t^ϵ by using the action functional for Y_t^ϵ and the fact that $u(x)$ is invertible.

Lemma 2.3. *Let $u(x)$ and $v(x)$ be strictly increasing functions as in Theorem 1.1. In addition, let $(v_n(x))_{n \in \mathbb{N}}$ be a sequence of strictly increasing functions, continuously differentiable with respect to $u(x)$ and converging to $v(x)$ at every continuity point of $v(x)$. Moreover, assume that $0 < c_2 \leq v'_n(x)$ for every n .*

Let $\phi : [0, T] \rightarrow \mathbb{R}$ be a continuous function in $[0, T]$, i.e. $\phi \in \mathcal{C}([0, T]; \mathbb{R})$. We introduce the functions $\sigma_\phi^n(t)$ by the formula

$$\sigma_\phi^n(t) = \int_0^t \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} ds \quad (2.5)$$

The functions $\sigma_\phi^n(t)$ can be regarded as functions of t or as functionals of ϕ . If $\Lambda(E_\phi) = 0$ then $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ exists uniformly in t on any finite time interval and independently of the choice of the sequence $(v_n)_{n \in \mathbb{N}}$. Moreover, it is continuous and non-decreasing in t . If $\Lambda(V_{d,\phi}) = 0$, then $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ is strictly increasing in t . We write

$$\sigma_\phi(t) = \int_0^t \left[\frac{1}{2} \frac{dv}{du}(\phi_s) \right]^{-1} ds = \lim_{n \rightarrow \infty} \sigma_\phi^n(t). \quad (2.6)$$

□

Lemma 2.4. *Let $\phi : [0, T] \rightarrow \mathbb{R}$ be a continuous function in $[0, T]$ such that $\sigma_\phi(t)$ is well defined for $t \in [0, T]$. Function $\gamma_\phi(t)$ is right continuous. Let us define $\gamma_\phi^-(t) = \lim_{s \rightarrow t^-} \gamma_\phi(s)$. For any $t \in [0, \sigma_\phi(T)]$ that is not a continuity point of $\gamma_\phi(t)$, the function $\phi(s)$ is constant for $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$.*

□

Lemma 2.5. *Let ϕ^n be a sequence of functions in $\mathcal{C}([0, T]; \mathbb{R})$ that converges to ϕ uniformly in $\mathcal{C}([0, T]; \mathbb{R})$. Under the assumptions of Theorem 1.1 for the functions v and u we have:*

- (i). *For any $t \in [0, T]$ we have that $\sigma_\phi(t) = \lim_{n \rightarrow \infty} \sigma_{\phi^n}(t)$. The convergence holds uniformly in t .*
- (ii). *For any $t \in [0, \sigma_\phi(T)]$ that is a continuity point of $\gamma_\phi(t)$ we have $\gamma_\phi(t) = \lim_{n \rightarrow \infty} \gamma_{\phi^n}(t)$.*
- (iii). *For any $t \in [0, \sigma_\phi(T)]$ we have $\phi(\gamma_\phi(t)) = \lim_{n \rightarrow \infty} \phi^n(\gamma_{\phi^n}(t))$.*

□

Proposition 2.6. *Let us define $X_t^\epsilon = u_{-1}[\sqrt{\epsilon}W_{\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)}]$, where $\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)$ is defined as in (2.2) with $\sqrt{\epsilon}W$ in place of W . Then, the infinitesimal generator of X_t^ϵ is $\epsilon DvDu$.*

□

Lemma 2.7. *The family $Y_t^\epsilon = \sqrt{\epsilon}W_{\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)}$, is exponentially tight in $\mathcal{C}([0, T]; \mathbb{R})$: for any $\alpha > 0$ and $\delta > 0$ there exists a compact $K_\alpha \subset \mathcal{C}([0, T]; \mathbb{R})$ such that*

$$\mathbb{P}(\rho_{0T}(Y^\epsilon, K_\alpha) \geq \delta) < \exp\{-\frac{\alpha}{\epsilon}\}$$

for $\epsilon > 0$ small enough.

□

Remark 2.8. *In what follows we will use Lemmas (2.4) and (2.5) with $\phi = u_{-1}(\psi)$, where ψ is a continuous function.*

□

Let us define now the functional

$$S_{0T}^Y(\psi) = \begin{cases} \frac{1}{2} \int_0^{\sigma_{u_{-1}(\psi)}(T)} \left| \frac{d\psi(\gamma_{u_{-1}(\psi)}(s))}{ds} \right|^2 ds, & \text{if } \psi(\gamma_{u_{-1}(\psi)}(s)) \text{ is absolutely continuous and } \psi_0 = u(x) \\ +\infty, & \text{for the rest of } \mathcal{C}([0, T]; \mathbb{R}). \end{cases} \quad (2.7)$$

Remark 1.3, Theorems 2.9 and 2.10 below imply that the action functional for the process $(Y_t^\epsilon)_{t \in [0, T]}$ on $\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$ is given by $\frac{1}{\epsilon} S_{0T}^Y(\psi)$. Theorem 2.9 discusses the standard properties of $S_{0T}^Y(\psi)$. In particular, $S_{0T}^Y(\psi)$ is lower semi-continuous in the sense of uniform convergence and the set $\Psi_s = \{\psi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}^Y(\psi) \leq s\}$ is compact. Theorem 2.10 is about the estimates for probabilities of large deviations. Then, as we mentioned before, Theorem 1.1 follows from these two theorems, Remark 1.3 and the well known contraction principle for large deviations.

Theorem 2.9. *Let u and v be two strictly increasing functions as in Theorem 1.1 and let $S_{0T}^Y(\psi)$ be defined by (2.7). Then*

- (i). *The functional $S_{0T}^Y(\psi)$ is lower semi-continuous in the sense of uniform convergence. Namely, if a sequence ψ^n converges uniformly to ψ in $\mathcal{C}([0, T]; \mathbb{R})$, then $S_{0T}^Y(\psi) \leq \liminf_{n \rightarrow \infty} S_{0T}^Y(\psi^n)$.*
- (ii). *The set $\Psi_s = \{\psi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}^Y(\psi) \leq s \text{ and } \psi(0) \text{ belongs to a compact subset of } \mathbb{R}\}$ is compact.*

□

Proof. (i). It is sufficient to consider the case when $S_{0T}^Y(\psi^n)$ has a finite limit. The proof follows directly from Lemma 2.5 and the fact that $\psi(\gamma_{u_{-1}(\psi)}(s))$ is absolutely continuous (see [16] page 75 and the proof of the corresponding property for the action functional of the Wiener process [8]).

(ii). Let $\psi \in \Psi_s$, i.e. $S_{0T}^Y(\psi) \leq s$. It is enough to prove that

a) $|\psi(t)| \leq C_0 < \infty$ for some constant C_0 uniformly in $t \in [0, T]$.

b) $|\psi(t+h) - \psi(t)| \leq g(h) \rightarrow 0$ as $h \rightarrow 0$ for some function $g(h)$ uniformly in $t \in [0, T]$.

Then we can conclude by the well known Ascoli-Arzelà theorem.

We have two cases: $\gamma_{u_{-1}(\psi)}(\cdot)$ is continuous at $\sigma_{u_{-1}(\psi)}(t) \in [0, \sigma_{u_{-1}(\psi)}(T)]$ and $\gamma_{u_{-1}(\psi)}(\cdot)$ is not continuous at $\sigma_{u_{-1}(\psi)}(t) \in [0, \sigma_{u_{-1}(\psi)}(T)]$ for $t \in [0, T]$.

Let $t \in [0, T]$ be such that $\gamma_{u_{-1}(\psi)}(\cdot)$ is continuous at $\sigma_{u_{-1}(\psi)}(t)$. In this case we certainly have $\gamma_{u_{-1}(\psi)}(\sigma_{u_{-1}(\psi)}(t)) = t$. Then, under the assumptions on the functions u and v , we easily see that

$$\begin{aligned} |\psi(t)| &\leq |\psi(t) - \psi(0)| + |\psi(0)| \\ &= \left| \int_0^{\sigma_{u_{-1}(\psi)}(t)} \frac{d\psi(\gamma_{u_{-1}(\psi)}(s))}{ds} ds \right| + |\psi(0)| \\ &\leq \sqrt{\sigma_{u_{-1}(\psi)}(t) 2S_{0T}^Y(\psi)} + |\psi(0)| \\ &\leq \sqrt{C_0 T} \sqrt{2s} + |\psi(0)| \end{aligned} \tag{2.8}$$

and similarly if $t, t+h \in [0, T]$ are such that $\gamma_{u_{-1}(\psi)}(\sigma_{u_{-1}(\psi)}(t)) = t$ and $\gamma_{u_{-1}(\psi)}(\sigma_{u_{-1}(\psi)}(t+h)) = t+h$, then

$$\begin{aligned} |\psi(t+h) - \psi(t)| &\leq \sqrt{2s} \sqrt{\sigma_{u_{-1}(\psi)}(t+h) - \sigma_{u_{-1}(\psi)}(t)} \\ &\leq \sqrt{2s} \sqrt{C_0 h}. \end{aligned}$$

Let $t \in [0, T]$ be such that $\gamma_{u_{-1}(\psi)}(\cdot)$ is not continuous at $\sigma_{u_{-1}(\psi)}(t)$. Since for any t we have $\gamma_{u_{-1}(\psi)}^-(\sigma_{u_{-1}(\psi)}(t)) \leq t \leq \gamma_{u_{-1}(\psi)}(\sigma_{u_{-1}(\psi)}(t))$, Lemma 2.4 implies that $\psi(t) = \psi(\gamma_{u_{-1}(\psi)}(\sigma_{u_{-1}(\psi)}(t)))$. Therefore, we have that the calculations in (2.8) remain valid in this case as well. This implies part a). For the equicontinuity part b) we can proceed in a similar way and prove that

$$|\psi(t+h) - \psi(t)| \leq \sqrt{2s} \sqrt{C_0 h}.$$

This concludes the proof of the theorem. \square

Theorem 2.10. *Let u and v be two strictly increasing functions as in Theorem 1.1 and let $S_{0T}^Y(\psi)$ be defined by (2.7). Then*

(i). *For any continuous $\psi : [0, T] \rightarrow \mathbb{R}$ and any $\delta, \eta > 0$ there exists an $\epsilon_0 > 0$ such that*

$$\mathbb{P}_x \left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < \delta \right) \geq \exp \left\{ -\frac{1}{\epsilon} (S_{0T}^Y(\psi) + \eta) \right\} \tag{2.9}$$

for $0 < \epsilon < \epsilon_0$.

(ii). *Let $s \in (0, \infty)$ and $\Psi_s = \{\psi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}^Y(\psi) \leq s\}$. For any $\delta, h > 0$ there exists an $\epsilon_0 > 0$ such that*

$$\mathbb{P}_x(\rho_{0T}(Y^\epsilon, \Psi_s) > \delta) \leq \exp \left\{ -\frac{1}{\epsilon} (s - \eta) \right\} \tag{2.10}$$

for $0 < \epsilon < \epsilon_0$. Here, $\rho_{0T}(\cdot, \cdot)$ is the uniform metric in $\mathcal{C}([0, T]; \mathbb{R})$.

□

Proof. Both statements are trivially true if ψ is such that $S_{0T}^Y(\psi) = \infty$. So, we assume that ψ is such that $S_{0T}^Y(\psi) < \infty$.

Throughout the proof of this Theorem we work with a sequence of functions $(v_n(x))_{n \in \mathbb{N}}$ as in the statement of Lemma 2.3. Lemma 2.3 guarantees that for ψ such that $\Lambda(E_{u_{-1}(\psi)}) = 0$ relation (2.6) holds with $\phi = u_{-1}(\psi)$. If the function ψ is such that $\Lambda(E_{u_{-1}(\psi)}) > 0$, then we consider a sequence $(v_n(x))_{n \in \mathbb{N}}$ such that, in addition to the previous requirements, relation (2.6) still holds (with the interpretation of $\sigma_{u_{-1}(\psi)}(t)$ given in the statement of Theorem 1.1). We claim that this restriction can be done without loss of generality. We leave the proof of this claim for the end and we continue with the proof of the Theorem.

(i). Let $n, N > 1$ be positive integers that will be chosen appropriately later on and recall the definition of the sequences $(\tau^n)_{n \in \mathbb{N}}$ and $(\sigma^n)_{n \in \mathbb{N}}$ by (2.1) and (2.5) respectively. We have

$$\begin{aligned} & \mathbb{P}_x\left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < \delta\right) \geq \\ & \geq \mathbb{P}_x\left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < \delta/N, \right. \\ & \quad \left. \sup_{0 \leq t \leq T} |\sqrt{\epsilon}W\left(\sigma_{u_{-1}(\psi)}(t) + [\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) - \sigma_{u_{-1}(\psi)}^n(t)] + \right. \right. \\ & \quad \left. \left. + [\tau_{u_{-1}(\sqrt{\epsilon}W)}(t) - \tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)] + [\sigma_{u_{-1}(\psi)}^n(t) - \sigma_{u_{-1}(\psi)}(t)]\right) - \psi(t)| < \delta\right) \end{aligned} \quad (2.11)$$

Note that the notation W_t and $W(t)$ are used equivalently.

Now by statement (i) of Theorem 2.1 we know that for every $\delta > 0$ and $\epsilon > 0$ and for n large enough, the following statement holds

$$\mathbb{P}_x\left(\sup_{0 \leq t \leq T} |\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) - \tau_{u_{-1}(\sqrt{\epsilon}W)}(t)| > \frac{\delta}{4}\right) \leq \exp\left\{-\frac{2}{\epsilon}S_{0T}^Y(\psi)\right\} \quad (2.12)$$

Moreover, the continuity of the function $\frac{dv_n}{du}$ and the fact that

$$\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) = \sigma_{u_{-1}(Y^\epsilon)}^n(t)$$

imply that for any $\delta_1 > 0$

$$\sup_{0 \leq t \leq T} |\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) - \sigma_{u_{-1}(\psi)}^n(t)| < \delta_1/2 \quad (2.13)$$

for trajectories $Y_t^\epsilon, 0 \leq t \leq T$, such that $\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < \delta/N$ with a large enough N that is independent of n .

By the choice of the approximating sequence $(v_n(x))_{n \in \mathbb{N}}$ and Lemma 2.3 we also have that

$$\sup_{0 \leq t \leq T} |\sigma_{u_{-1}(\psi)}^n(t) - \sigma_{u_{-1}(\psi)}(t)| < \delta_1/2 \quad (2.14)$$

for n large enough.

Furthermore, for a one dimensional Wiener process W_t we have

$$\begin{aligned}
& \mathbb{P}_x \left(\sqrt{\epsilon} \max_{0 \leq t \leq T} \max_{|s| \leq \delta_1, t+s \geq 0} |W_{t+s} - W_s| > \frac{\delta}{4} \right) \\
& \leq \sum_{k=1}^{\left\lceil \frac{T}{\delta_1} \right\rceil + 1} \mathbb{P}_x \left(\sqrt{\epsilon} \max_{0 \leq s \leq 2\delta_1} \left| W_{\frac{kT}{\delta_1} + s} - W_{\frac{kT}{\delta_1}} \right| > \frac{\delta}{4} \right) \\
& \leq \left(\frac{T}{\delta_1} + 1 \right) \mathbb{P}_x \left(\sqrt{\epsilon} \max_{0 \leq s \leq 2\delta_1} |W_s| > \frac{\delta}{4} \right) \\
& \leq \frac{T+1}{\delta_1} \exp\left\{-\frac{\delta^2}{4\epsilon\delta_1}\right\} \\
& \leq \exp\left\{-\frac{2}{\epsilon} S_{0T}^Y(\psi)\right\}
\end{aligned} \tag{2.15}$$

for $\delta_1 = \delta/10S_{0T}^Y(\psi)$ and $\epsilon > 0$ small enough.

Combining now relations (2.11)-(2.15) and Lemma 2.4 we get

$$\begin{aligned}
& \mathbb{P}_x \left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < \delta \right) \geq \\
& \geq \mathbb{P}_x \left(\sup_{0 \leq t \leq T} |\sqrt{\epsilon}W(\sigma_{u_{-1}(\psi)}(t)) - \psi(t)| < \frac{\delta}{4} \right) - 3 \exp\left\{-\frac{2}{\epsilon} S_{0T}^Y(\psi)\right\} \\
& = \mathbb{P}_x \left(\sup_{0 \leq t \leq \sigma_{u_{-1}(\psi)}(T)} |\sqrt{\epsilon}W(t) - \psi(\gamma_{u_{-1}(\psi)}(t))| < \frac{\delta}{4} \right) - 3 \exp\left\{-\frac{2}{\epsilon} S_{0T}^Y(\psi)\right\} \\
& \geq C_0 \exp\left\{-\frac{1}{\epsilon} (S_{0T}^Y(\psi) + \eta)\right\}
\end{aligned} \tag{2.16}$$

for ϵ small enough. In the last inequality we used the well known formula for the action functional of the Gaussian process $\sqrt{\epsilon}W(t)$ on the function $\psi(\gamma_{u_{-1}(\psi)}(t))$ for $0 \leq t \leq \sigma_{u_{-1}(\psi)}(T)$.

(ii). By Lemma 2.7 we know that Y_t^ϵ is exponential tight. Hence for $\alpha = 2s+1$ we have

$$\mathbb{P}(\rho_{0T}(Y^\epsilon, K_{2s+1}) \geq \delta) < \exp\left\{-\frac{2s+1}{\epsilon}\right\}$$

We have

$$\begin{aligned}
& \mathbb{P}_x(\rho_{0T}(Y^\epsilon, \Psi_s) > \delta) = \mathbb{P}_x(\rho_{0T}(Y^\epsilon, \Psi_s) > \delta, \rho_{0T}(Y^\epsilon, K_{2s+1}) < \delta) + \\
& \quad + \mathbb{P}_x(\rho_{0T}(Y^\epsilon, \Psi_s) > \delta, \rho_{0T}(Y^\epsilon, K_{2s+1}) > \delta) \leq \\
& \leq \mathbb{P}_x(\rho_{0T}(Y_s^\epsilon, K_{2s+1}) < \delta) + \exp\left\{-\frac{2s+1}{\epsilon}\right\}
\end{aligned} \tag{2.17}$$

Let now $\psi \in K_{2s+1} \setminus \Psi_s$. Recall that $Y_t^\epsilon = \sqrt{\epsilon}W\left(\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)\right)$. Hence, we have

$$\begin{aligned}
& \mathbb{P}_x \left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < 2\delta \right) = \\
& = \mathbb{P}_x \left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < 2\delta, \right. \\
& \quad \sup_{0 \leq t \leq T} |\sqrt{\epsilon}W\left(\sigma_{u_{-1}(\psi)}(t) + [\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) - \sigma_{u_{-1}(\psi)}^n(t)] + \right. \\
& \quad \left. \left. + [\tau_{u_{-1}(\sqrt{\epsilon}W)}(t) - \tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)] + [\sigma_{u_{-1}(\psi)}^n(t) - \sigma_{u_{-1}(\psi)}(t)]\right) - \psi(t)| < 2\delta \right)
\end{aligned} \tag{2.18}$$

Using (2.12)-(2.15) and Lemma 2.4, the latter implies that for n large enough and δ small enough we have

$$\begin{aligned}
& \mathbb{P}_x\left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - \psi(t)| < 2\delta\right) \leq \\
& \leq \mathbb{P}_x\left(\sup_{0 \leq t \leq T} |\sqrt{\epsilon}W(\sigma_{u_{-1}(\psi)}(t)) - \psi(t)| < 8\delta\right) + 3\exp\left\{-\frac{2s+1}{\epsilon}\right\} \\
& = \mathbb{P}_x\left(\sup_{0 \leq t \leq \sigma_{u_{-1}(\psi)}(T)} |\sqrt{\epsilon}W(t) - \psi(\gamma_{u_{-1}(\psi)}(t))| < 8\delta\right) + 3\exp\left\{-\frac{2s+1}{\epsilon}\right\} \\
& \leq \exp\left\{-\frac{1}{\epsilon}(S_{0T}^Y(\psi) - \eta)\right\} + 3\exp\left\{-\frac{2s+1}{\epsilon}\right\} \\
& \leq C_0 \exp\left\{-\frac{1}{\epsilon}(s - \eta)\right\}
\end{aligned} \tag{2.19}$$

for ϵ small enough. In the last inequality we used the well known formula for the action functional of the Gaussian process $\sqrt{\epsilon}W(t)$ on the function $\psi(\gamma_{u_{-1}(\psi)}(t))$ for $0 \leq t \leq \sigma_{u_{-1}(\psi)}(T)$ and that for $\psi \in K_{2s+1} \setminus \Psi_s$ we have $S_{0T}^Y(\psi) \geq s$.

Let now ψ^i for $i \in \{1, \dots, N\}$ be a finite δ -net of $K_{2s+1} \setminus \Psi_s$. Then (2.17) and (2.19) imply that

$$\mathbb{P}_x(\rho_{0T}(Y_s^\epsilon, \Psi_s) > \delta) \leq C_0 \exp\left\{-\frac{1}{\epsilon}(s - \eta)\right\} \tag{2.20}$$

for ϵ small enough. This concludes the proof of part (ii) of the Theorem.

It remains to prove the claim made in the beginning of the proof. Let us pick a point $z \in (U \cup V) \setminus V_d$ and let us write for notational convenience $\phi = u_{-1}(\psi)$. Denote $E_\phi^z = \{t \in [0, T] : \phi_t = z\}$. Essentially, we have two cases

- (i). Assume that E_ϕ^z is an interval, for example $E_\phi^z = [t_0, t_1] \subset [0, T]$. We will have that $\dot{\phi}_t = 0$ for every $t \in (t_0, t_1)$. Then, it is easy to see that $S_{E_\phi^z}(\phi) = S_{t_0 t_1}(\phi) = 0$ (e.g., from expression (1.11)).
- (ii). Assume that E_ϕ^z is not an interval. Then, one can use Theorem A.6.3 in [1] or Problem 11 on pages 334–335 of [2] to claim that the Lebesgue measure of the set $\{t \in [0, T] : \phi_t = z, \dot{\phi}_t \neq 0\}$ is zero (due to absolute continuity).

Hence, in either case we have that $S_{E_\phi^z}(\phi) = 0$. In other words, even though, for the case $\Lambda(E_\phi) > 0$, the limit of $\sigma_\phi^n(t)$ as $n \rightarrow \infty$ is affected by the approximating sequence $(v_n(x))_{n \in \mathbb{N}}$, the corresponding action functional is not. Thus, we can make the convention that was made in part (i) of Theorem 1.1.

□

We conclude this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 2.3 is essentially statement (i) of Theorem 1.1.

As far as statement (ii) of Theorem 1.1 is concerned, we have the following. By Remark 1.3 and Theorems 2.9 and 2.10 we have that $\frac{1}{\epsilon}S_{0T}^Y(\psi)$ is the action functional for the process $(Y_t^\epsilon)_{t \in [0, T]}$ on

$\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$. Then by the contraction principle we have that the action functional for the process $(X_t^\epsilon)_{t \in [0, T]}$ on $\mathcal{C}([0, T]; \mathbb{R})$ as $\epsilon \downarrow 0$ is given by $\frac{1}{\epsilon} S_{0T}(\phi)$, where

$$\begin{aligned} S_{0T}(\phi) &= \inf \{ S_{0T}^Y(\psi) : \psi = u(\phi) \} \\ &= S_{0T}^Y(u(\phi)) \end{aligned}$$

The compactness of the set $\Phi_s = \{ \phi \in \mathcal{C}([0, T]; \mathbb{R}) : S_{0T}(\phi) \leq s \}$ and the lower semicontinuity of $S_{0T}(\phi)$ follows immediately from the corresponding statements for Ψ_s and $S_{0T}^Y(\psi)$. \square

3 Proof of auxiliary results

In this section we prove Lemma 2.3, Lemma 2.4, Lemma 2.5, Proposition 2.6 and Lemma 2.7.

Proof of Lemma 2.3. A lemma similar to this one is stated without proof in [20]. Here, we provide for completeness a sketch of the proof for our case of interest.

Let $\phi : [0, T] \rightarrow \mathbb{R}$ be a continuous function in $[0, T]$, i.e. $\phi \in \mathcal{C}([0, T]; \mathbb{R})$. Recall that the functions $\sigma_\phi^n(t)$ are defined by the formula

$$\sigma_\phi^n(t) = \int_0^t \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} ds.$$

It is easy to see now, that it is enough to prove that $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ exists for any $t \in [0, T]$ independently of the choice of the sequence $(v_n)_{n \in \mathbb{N}}$. Then, uniformity follows from the latter and the fact that the first derivatives of the functions $\sigma_\phi^n(t)$ are bounded uniformly in n and $t \in [0, T]$. The assumptions on the functions u and v_n guarantee the boundedness of the first derivatives of $\sigma_\phi^n(t)$.

It is clear that $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ exists, independently of the choice of the sequence $(v_n)_{n \in \mathbb{N}}$, if the Lebesgue measure of $V_{d,\phi}$ is zero, i.e. $\Lambda(V_{d,\phi}) = 0$. In this case, the $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ is continuous and strictly increasing function of t .

Hence, it remains to consider the case $\Lambda(V_{d,\phi}) > 0$. It is enough to prove that for any $\epsilon > 0$ there is a $n_0(\epsilon) > 0$ such that

$$\left| \int_{V_{d,\phi}} \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} - \left[\frac{1}{2} \frac{dv_m}{du}(\phi_s) \right]^{-1} ds \right| < \epsilon \text{ for any } n, m \geq n_0(\epsilon).$$

We write

$$\begin{aligned} & \left| \int_{V_{d,\phi}} \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} - \left[\frac{1}{2} \frac{dv_m}{du}(\phi_s) \right]^{-1} ds \right| \leq \\ & \leq \left| \int_{V_{d,\phi} \setminus U_\phi} \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} - \left[\frac{1}{2} \frac{dv_m}{du}(\phi_s) \right]^{-1} ds \right| + \\ & + \left| \int_{V_{d,\phi} \cap U_\phi} \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} - \left[\frac{1}{2} \frac{dv_m}{du}(\phi_s) \right]^{-1} ds \right| \end{aligned}$$

If $\Lambda(V_{d,\phi} \cap U_\phi) = 0$, then the second term in the inequality above is zero and it is easily seen that the first term can be made arbitrarily small for n, m large enough.

If, on the other hand, $\Lambda(U_\phi \cap V_{d,\phi}) > 0$, then we may define

$$\lim_{n \rightarrow \infty} \int_{U_\phi \cap V_{d,\phi}} \left[\frac{1}{2} \frac{dv_n}{du}(\phi_s) \right]^{-1} ds = 0$$

and the result follows. Therefore, in the case $\Lambda(E_\phi) = 0$, the $\lim_{n \rightarrow \infty} \sigma_\phi^n(t)$ exists and the limit is independent of the approximating sequence $(v_n)_{n \in \mathbb{N}}$. Finally, it is easily seen that the limit is non decreasing and continuous in t . \square

Proof of Lemma 2.4. It is clear that $\gamma_\phi(t)$ is right continuous. Moreover, it is easy to see that continuity of $\sigma_\phi(\cdot)$ implies that $\sigma_\phi(\gamma_\phi^-(t)) = \sigma_\phi(\gamma_\phi(t))$. This implies that $\phi(s) \in V_d$ almost everywhere in $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$. Recall that V_d is the set of discontinuity points for function $v(x)$.

Let now $x_0 \in V_d$ such that $\phi(\gamma_\phi^-(t)) = x_0$. Define

$$s_o = \sup\{s : s \in [\gamma_\phi^-(t), \gamma_\phi(t)], \phi(s) = x_0, \phi(\rho) \notin V_d \setminus \{x_0\} \text{ for all } \rho < s\}$$

If $s_o = \gamma_\phi(t)$ then $\phi(s)$ is constant almost everywhere in $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$. Therefore, $\phi(s)$ is constant everywhere in $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$ since $\phi(s)$ is continuous.

Assume that there is some $x_1 \in V_d$ with $x_1 \neq x_0$ such that $\phi(s) = x_1$ for some $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$. In particular, define

$$s_1 = \inf\{s : s \in (s_o, \gamma_\phi(t)], \phi(s) \in V_d \setminus \{x_0\}\}.$$

We write $\phi(s_1) = x_1$. Of course, if $s_0 = s_1$ then we have a contradiction since $\phi(s_0) = x_0$ and $\phi(s_1) = x_1$. So, we assume that $s_0 < s_1$. In this case we clearly have that $\sigma_\phi(s_0) < \sigma_\phi(s_1)$. However, since $[s_0, s_1] \subset [\gamma_\phi^-(t), \gamma_\phi(t)]$ and $\sigma_\phi(\cdot)$ is non decreasing and continuous, the latter clearly contradicts $\sigma_\phi(\gamma_\phi^-(t)) = \sigma_\phi(\gamma_\phi(t))$. Hence, such an x_1 does not exist. The latter implies that $\phi(s)$ is constant almost everywhere in $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$. Therefore, $\phi(s)$ is constant everywhere in $s \in [\gamma_\phi^-(t), \gamma_\phi(t)]$ since $\phi(s)$ is continuous. \square

Proof of Lemma 2.5. Let ϕ^n be a sequence of functions in $\mathcal{C}([0, T]; \mathbb{R})$ that converges to ϕ uniformly in $\mathcal{C}([0, T]; \mathbb{R})$. We only prove parts (ii) and (iii). Part (i) is easily seen to hold by the uniform convergence of ϕ^n to ϕ .

Let $t_* \in [0, \sigma_\phi(T)]$ be a continuity point of $\gamma_\phi(t)$. Of course, $\gamma_\phi(t)$ can only have countable many points of discontinuity.

Let $s_* \in [0, T]$ be such that $t_* = \sigma_\phi(s_*)$. Such an s_* exists because $\sigma_\phi(s)$ is continuous. By part (i) we have that for any $\epsilon > 0$ there is an $n_0(\epsilon) \in \mathbb{N}$ such that

$$|\sigma_{\phi^n}(s) - \sigma_\phi(s)| < \epsilon$$

for every $s \in [0, T]$ and $n \geq n_0(\epsilon)$.

The latter and the fact that $\gamma_{\phi^n}(t)$ is non-decreasing give us

$$\gamma_{\phi^n}(\sigma_{\phi^n}(s_*) - \epsilon) \leq \gamma_{\phi^n}(\sigma_\phi(s_*)) \leq \gamma_{\phi^n}(\sigma_{\phi^n}(s_*) + \epsilon)$$

For $n \geq n_0(\epsilon)$ we have

$$\begin{aligned} \gamma_{\phi^n}(\sigma_{\phi^n}(s_*) + \epsilon) &= \inf\{s : \sigma_{\phi^n}(s) > \sigma_{\phi^n}(s_*) + \epsilon\} \\ &\leq \inf\{s : \sigma_{\phi}(s) > \sigma_{\phi^n}(s_*) + 2\epsilon\} \\ &\leq \inf\{s : \sigma_{\phi}(s) > \sigma_{\phi}(s_*) + 3\epsilon\} \\ &= \gamma_{\phi}(\sigma_{\phi}(s_*) + 3\epsilon) \end{aligned}$$

Likewise, for n large enough

$$\gamma_{\phi^n}(\sigma_{\phi^n}(s_*) - \epsilon) \geq \gamma_{\phi}(\sigma_{\phi}(s_*) - 3\epsilon)$$

Therefore, for n large enough, we have

$$\gamma_{\phi}(\sigma_{\phi}(s_*) - 3\epsilon) \leq \gamma_{\phi^n}(\sigma_{\phi}(s_*)) \leq \gamma_{\phi}(\sigma_{\phi}(s_*) + 3\epsilon) \quad (3.1)$$

Therefore, (3.1) implies that

$$\gamma_{\phi^n}(\sigma_{\phi}(s_*)) \rightarrow \gamma_{\phi}(\sigma_{\phi}(s_*)) \text{ as } n \rightarrow \infty,$$

or in other words

$$\gamma_{\phi^n}(t_*) \rightarrow \gamma_{\phi}(t_*) \text{ as } n \rightarrow \infty,$$

which concludes the proof of part (ii) of the lemma.

Lastly, we prove part (iii) of the lemma. Let $t \in [0, \sigma_{\phi}(T)]$. We write

$$\begin{aligned} |\phi^n(\gamma_{\phi^n}(t)) - \phi(\gamma_{\phi}(t))| &\leq |\phi^n(\gamma_{\phi}(t)) - \phi(\gamma_{\phi}(t))| \\ &\quad + |\phi^n(\gamma_{\phi^n}(t)) - \phi^n(\gamma_{\phi}(t))| \end{aligned} \quad (3.2)$$

The uniform convergence of ϕ^n to ϕ guarantees that the first term in the right hand side of (3.2) can be made arbitrarily small for n large enough. Moreover, part (ii), guarantees that the second term can be arbitrarily small provided that t is a continuity point of $\gamma_{\phi}(\cdot)$. Hence, it is enough to consider the case where t is not a continuity point of $\gamma_{\phi}(\cdot)$. We claim that the following two statements hold.

- a) For every $\epsilon > 0$ there is a $n_0(\epsilon) > 0$ such that for every $t \in [0, \sigma_{\phi}(T)]$ and for every $n > n_0(\epsilon)$ we have that

$$\gamma_{\phi^n}(t) \in [\gamma_{\phi}(t - \epsilon), \gamma_{\phi}(t + \epsilon)].$$

- b) The function $\phi(s)$ is constant for $s \in [\gamma_{\phi}^-(t), \gamma_{\phi}(t)]$, where we set $\gamma_{\phi}^-(t) = \lim_{s \rightarrow t^-} \gamma_{\phi}(s)$.

These statements together with the uniform convergence ϕ^n to ϕ guarantee that the second term in the right hand side of (3.2) can be made arbitrarily small for n large enough even if t is not a continuity point of $\gamma_{\phi}(\cdot)$. Hence, it remains to prove the claim. Part a) follows by an argument similar to the one that was used in the proof of part (ii) of this lemma (see (3.1)) and part b) is Lemma 2.4.

This concludes the proof of the lemma. \square

Proof of Proposition 2.6. Recall that $X_t^\epsilon = u_{-1}[\sqrt{\epsilon}W_{\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)}]$, where $\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)$ is defined as in (2.2) with $\sqrt{\epsilon}W$ in place of W . Let us also define $\hat{X}_t = u_{-1}[W_{\hat{\tau}_{u_{-1}(W)}(t)}]$, where $\hat{\tau}_{u_{-1}(W)}(t)$ is defined similarly to (2.2). Then, we easily see that

$$\begin{aligned} t &= \int_0^{\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(\sqrt{\epsilon}W_s)) ds \\ &= \frac{1}{\epsilon} \int_0^{\epsilon \tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(W_s)) ds. \end{aligned}$$

On the other hand, it is also true that

$$t = \frac{1}{\epsilon} \int_0^{\hat{\tau}_{u_{-1}(W)}^n(\epsilon t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(W_s)) ds.$$

The latter imply that

$$\int_0^{\epsilon \tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(W_s)) ds = \int_0^{\hat{\tau}_{u_{-1}(W)}^n(\epsilon t)} \frac{1}{2} \frac{dv_n}{du}(u_{-1}(W_s)) ds$$

Taking into account that $\tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t)$ and $\hat{\tau}_{u_{-1}(W)}^n(t)$ are strictly increasing in t and that $\frac{dv_n}{du}$ is strictly positive, we get that almost surely

$$\epsilon \tau_{u_{-1}(\sqrt{\epsilon}W)}^n(t) = \hat{\tau}_{u_{-1}(W)}^n(\epsilon t).$$

The latter implies that

$$\begin{aligned} X_t^\epsilon &= u_{-1}[\sqrt{\epsilon}W_{\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)}] \\ &= u_{-1}[W_{\hat{\tau}_{u_{-1}(W)}(\epsilon t)}] \\ &= \hat{X}_{\epsilon t}. \end{aligned} \tag{3.3}$$

Let now I be an interval in \mathbb{R} and T_I and \hat{T}_I be the exit times for X_t^ϵ , \hat{X}_t from I respectively. Then using (3.3), the infinitesimal generator of X_t^ϵ is

$$\begin{aligned} \lim_{dI \rightarrow 0} \frac{\mathbb{E}_x f(X_{T_I}^\epsilon) - f(x)}{\mathbb{E}_x T_I} &= \lim_{dI \rightarrow 0} \frac{\mathbb{E}_x f(\hat{X}_{\hat{T}_I}) - f(x)}{\mathbb{E}_x \hat{T}_I} \frac{\mathbb{E}_x \hat{T}_I}{\mathbb{E}_x T_I} \\ &= \epsilon D_v D_u, \end{aligned}$$

where dI is the length of I . This concludes the proof of the proposition. \square

Proof of Lemma 2.7. The result can be easily derived by the representation $Y_t^\epsilon = \sqrt{\epsilon}W_{\tau_{u_{-1}(\sqrt{\epsilon}W)}(t)}$ and Theorem 4.1 of [4]. \square

4 Generalized reaction-diffusion equations and some results on wave front propagation

In this section we discuss reaction-diffusion equations governed by a generalized elliptic operator $D_v D_u$. We will refer to them as generalized reaction diffusion equations. We apply Theorem 1.1 to the problem of wave front propagation for these type of reaction-diffusion equations in the case where the non-linear term is of K-P-P type.

Let $D_v D_u$ be the operator introduced in the introduction. For $f \in \mathcal{D}(D_v D_u)$, i.e. for functions that belong to the domain of definition of the $D_v D_u$ operator, consider the following reaction diffusion equation

$$\begin{aligned} f_t(t, x) &= D_v D_u f(t, x) + c(x, f(t, x))f(t, x) \\ f(0, x) &= g(x) \end{aligned} \quad (4.1)$$

We shall consider the generalized solution to (4.1). We define the operator

$$Af = -f_t + D_v D_u f.$$

As it is well known, there exists a corresponding Markov family $Y_s = (t - s, X_s)$ in the state space $(-\infty, T] \times \mathbb{R}, T > 0$. Here X_s is the strong Markov process governed by the operator $D_v D_u$. Moreover, we define $f(t, x) = g(x)$ for $t \leq 0$. Using the Feynman-Kac formula, the solution to this problem may be written as follows:

$$f(t, x) = \mathbb{E}_x g(X_t) e^{\int_0^t c(X_s, f(t-s, X_s)) ds} \quad (4.2)$$

We shall call the solution to equation (4.2) the generalized solution to equation (4.1). Throughout this section, we will make the following assumption.

Assumption 4.1. *The function $c(x, f)$ is uniformly bounded in all arguments, continuous in x and Lipschitz continuous in f . The initial profile $g(x)$ is a bounded, nonnegative function that can have at most a finite number of simple discontinuities.*

□

One can prove, via the standard method of successive approximations, that under the aforementioned assumption, there exists a unique generalized solution for the problem (4.1). Namely, the equation (4.2) has a unique solution (see chapter 5 of [5] for more details).

Generalized reaction diffusion equations, like (4.1), can appear in applications as, for example, the limit of a family of standard reaction-diffusion equations.

Let us demonstrate this in a simple case. Consider the family of problems

$$\begin{aligned} f_t^n(t, x) &= L_n f^n(t, x) + c(x, f^n(t, x))f^n(t, x) \\ f^n(0, x) &= g(x) \end{aligned} \quad (4.3)$$

where L_n is a family of standard second order elliptic operators

$$L_n f(x) = \frac{1}{2} a_n(x) \frac{d^2 f(x)}{dx^2} + b_n(x) \frac{df(x)}{dx}. \quad (4.4)$$

Assume that the limits of the coefficients $a_n(x)$ and $b_n(x)$ are discontinuous as follows

$$\lim_{n \rightarrow \infty} a_n(x) = a(x) = \begin{cases} a_+(x), & x > 0 \\ a_-(x), & x < 0. \end{cases}$$

and

$$\lim_{n \rightarrow \infty} b_n(x) = b(x) = \begin{cases} b_+(x), & x > 0 \\ b_-(x), & x < 0. \end{cases}$$

where $a(x)$ and $b(x)$ may not be defined or be discontinuous at $x = 0$. Define

$$u_n(x) = \int_0^x e^{-\int_0^y \frac{2b_n(z)}{a_n(z)} dz} dy \quad \text{and} \quad v_n(x) = \int_0^x \frac{2}{a_n(y)} e^{\int_0^y \frac{2b_n(z)}{a_n(z)} dz} dy.$$

We observe that $D_{v_n} D_{u_n} f = L_n f$. Let X_t^n be the one dimensional Markov process with infinitesimal generator L_n and let $\tau^n(-\delta, \delta) = \inf\{t : X_t^n \notin (-\delta, \delta)\}$. Define the quantities

$$\begin{aligned} P_r &= \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \mathbb{P}_x(X_{\tau^n} = \delta) = \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \frac{u_n(x) - u_n(-\delta)}{u_n(\delta) - u_n(-\delta)} \\ P_l &= \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \mathbb{P}_x(X_{\tau^n} = -\delta) = \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \frac{u_n(\delta) - u_n(x)}{u_n(\delta) - u_n(-\delta)} \\ \kappa &= \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}_x \frac{1}{\delta} \tau^n(-\delta, \delta). \end{aligned}$$

The function $m_n(x) = \mathbb{E}_x \tau^n(-\delta, \delta)$ is solution to the equation $D_{v_n} D_{u_n} m_n(x) = -1$ with boundary conditions $m_n(-\delta) = m_n(\delta) = 0$.

If $P_r = P_l = \frac{1}{2}$ and $\kappa = 0$, then the limit (in distribution) of X_t^n behaves locally like a Wiener process. But, of course, this is not the case in general. Define the functions

$$\begin{aligned} u(x) &= \begin{cases} \frac{1}{P_r} \int_0^x e^{-\int_0^y \frac{2b(z)}{a(z)} dz} dy, & x \geq 0 \\ \frac{1}{P_l} \int_0^x e^{-\int_0^y \frac{2b(z)}{a(z)} dz} dy, & x < 0. \end{cases} \\ v(x) &= \begin{cases} \kappa + P_r \int_0^x \frac{2}{a(y)} e^{\int_0^y \frac{2b(z)}{a(z)} dz} dy, & x \geq 0 \\ P_l \int_0^x \frac{2}{a(y)} e^{\int_0^y \frac{2b(z)}{a(z)} dz} dy, & x < 0. \end{cases} \end{aligned}$$

and assume that P_r, P_l, κ and that the limit $\lim_{n \rightarrow \infty} e^{-\int_0^y \frac{2b_n(z)}{a_n(z)} dz}$ exists for all $y \in \mathbb{R}$. It is easy to see that

$$\begin{aligned} u(x) &= \lim_{n \rightarrow \infty} u_n(x) \text{ for every } x \in \mathbb{R} \\ v(x) &= \lim_{n \rightarrow \infty} v_n(x) \text{ for every } x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

Then, it can be shown (see [12] for more details) that

$$\lim_{n \rightarrow \infty} f^n(t, x) = f(t, x),$$

where $f_n(t, x)$ and $f(t, x)$ are the generalized solutions to (4.3) and (4.1) respectively. In this case, the domain of definition of the $D_v D_u$ operator is

$$\begin{aligned} \mathcal{D}(D_v D_u) = \{ & f : f \in \mathcal{C}_c(\mathbb{R}), \text{ with } f_x, f_{xx} \in \mathcal{C}(\mathbb{R} \setminus \{0\}), \\ & P_r f'_+(0) - P_l f'_-(0) = \kappa D_v D_u f(0) \text{ and} \\ & D_v D_u f(0) = \lim_{x \rightarrow 0^+} D_v D_u f(x) = \lim_{x \rightarrow 0^-} D_v D_u f(x) \}. \end{aligned}$$

Let us study now the problem of wave front propagation for the following equation. For $f \in \mathcal{D}(D_v D_u)$ consider the generalized solution to the following reaction diffusion equation

$$\begin{aligned} f_t^\epsilon(t, x) &= \epsilon D_v D_u f^\epsilon(t, x) + \frac{1}{\epsilon} c(x, f^\epsilon(t, x)) f^\epsilon(t, x) \\ f^\epsilon(0, x) &= g(x) \end{aligned} \tag{4.5}$$

For brevity, we consider the initial profile of (4.5) to be given by $g(x) = \chi_{x \leq 0}$, where $\chi_{x \leq 0}$ is the characteristic function of the set $\{x : x \leq 0\}$. Moreover, the non linear function $c(x, f)$ is assumed to be of Kolmogorov-Petrovskii-Piskunov (K-P-P) type, i.e. it is Lipschitz continuous in $f \in \mathbb{R}$, positive for $f < 1$, negative for $f > 1$ and $c(x) = c(x, 0) = \max_{0 \leq f \leq 1} c(x, f)$. Generalized reaction diffusion equations that have a K-P-P type nonlinear term are called K-P-P generalized reaction diffusion equations.

It is not difficult to see that the classical results of Freidlin [5] on wave front propagation of K-P-P reaction diffusion equations hold in this case as well. Let us define

$$W(t, x) = \sup \left\{ \int_0^t c(\phi_s) ds - S_{0t}(\phi) : \phi \in \mathcal{C}_{0,t}, \phi_0 = x, \phi_t \leq 0 \right\}. \tag{4.6}$$

where $c(x) = c(x, 0) = \max_{0 \leq f \leq 1} c(x, f)$ and $S_{0t}(\phi)$, defined by (1.10), is the action functional for the Markov process X_t^ϵ whose infinitesimal generator is $\epsilon D_v D_u$.

We say that condition (N) is satisfied if for any $t > 0$ and $(t, x) \in \{(t, x) : W(t, x) = 0\}$:

$$\begin{aligned} W(t, x) = \sup \left\{ \int_0^t c(\phi_s) ds - S_{0t}(\phi) : \phi_0 = x, \phi_t \leq 0, \right. \\ \left. (t - s, \phi_s) \in \{(t, x) : W(t, x) < 0\} \right\}. \end{aligned}$$

Theorem 4.2. (Freidlin [5]). *Let $f^\epsilon(t, x)$ be the unique generalized solution to (4.5). Then, under condition (N) we have:*

$$\lim_{\epsilon \downarrow 0} f^\epsilon(t, x) = \begin{cases} 1, & W(t, x) > 0 \\ 0, & W(t, x) < 0. \end{cases} \tag{4.7}$$

The convergence is uniform on every compactum lying in the region $\{(t, x) : t > 0, x \in \mathbb{R}, W(t, x) > 0\}$ and $\{(t, x) : t > 0, x \in \mathbb{R}, W(t, x) < 0\}$ respectively.

Hence, the equation $W(t, x) = 0$ defines the position of the interface (wavefront) between areas where f^ϵ (for $\epsilon > 0$ small enough) is close to 0 and to 1. Moreover, $W(t, x)$ is a continuous function, increasing in t .

We shall consider a simple example that illustrates the applicability of Theorem 1.1. Assume, for brevity, that

$$\begin{aligned} u(x) &= x \\ v(x) &= \begin{cases} Ax, & x < x_1 \\ \kappa + Ax, & x_1 \leq x \leq x_2 \\ \kappa + Ax + B(x - x_2), & x \geq x_2. \end{cases} \\ c(x) &= c(x, 0) = c = \text{constant}, \end{aligned} \quad (4.8)$$

where κ, A and B are positive constants and $0 < x_1 < x_2$. Of course κ is the jump of the function $v(x)$ at $x = x_1$. Moreover, $v(x)$ has a corner point at $x = x_2$.

The process X_t^c that is governed by the operator $\epsilon D_v D_u$ is a time changed Wiener process with delay at $x = x_1$.

We shall derive the position of the wave front for this simple case.

It is clear that inside the half lines and line segments $\{x < x_1\}, \{x_1 < x < x_2\}$ and $\{x > x_2\}$ the process X_t^c that is governed by the operator $\epsilon D_v D_u$ behaves like a standard Wiener process. Hence, the extremals ϕ of the variational problem (4.6) for the functional $R_{0t}(\phi) = ct - S_{0t}(\phi)$ are line segments. Moreover, clearly, condition (N) holds.

The position of the wave front (interface) for any couple (t, x) is given by the equation $W(t, x) = 0$. Let $t_* = t_*(x)$ satisfy the equation $W(t_*(x), x) = 0$. Such a $t_*(x)$ is defined in a unique way.

For $x \in [0, x_1]$ the position of the wave front is

$$W(t_*, x) = 0 \Rightarrow ct_* - \frac{A}{4} \frac{x^2}{t_*} = 0 \Rightarrow t_*(x) = \sqrt{\frac{A}{4c}} x \quad (4.9)$$

For $x \in [x_1, x_2]$ the position of the wave front is as follows. Assume that $0 \leq \mu_0 \leq \mu_1 \leq t_*$ and that for $t \in [0, \mu_0]$ and for $t \in [\mu_1, t_*]$ the function ϕ is linear. For $t \in [\mu_0, \mu_1]$ we assume that $\phi(t) = x_1$. Straightforward algebra shows that

$$\begin{aligned} \sigma_\phi(t) &= \begin{cases} \frac{2}{A}t, & 0 \leq t \leq \mu_0 \\ \frac{2}{A}\mu_0, & \mu_0 \leq t < \mu_1 \\ \frac{2}{A}(t - \mu_1 + \mu_0), & \mu_1 \leq t \leq t_*. \end{cases} \\ \phi(\gamma_\phi(t)) &= \begin{cases} \frac{x_1 - x}{\mu_0} \frac{A}{2}t + x, & 0 \leq t \leq \frac{2}{A}\mu_0 \\ -\frac{x_1}{t_* - \mu_1} \left(\frac{A}{2}t + \mu_1 - \mu_0 \right) + \frac{x_1 t_*}{t_* - \mu_1}, & \frac{2}{A}\mu_0 \leq t \leq \frac{2}{A}(t_* - \mu_1 + \mu_0). \end{cases} \end{aligned}$$

Therefore, we get

$$\begin{aligned} W(t_*, x) = 0 &\Rightarrow ct_* - \inf_{0 \leq \mu_0 \leq \mu_1 \leq t_*} \left\{ \frac{A}{4} \frac{(x - x_1)^2}{\mu_0} + \frac{A}{4} \frac{x_1^2}{t_* - \mu_1} \right\} = 0 \\ &\Rightarrow t_*(x) = \sqrt{\frac{A}{4c}} x \end{aligned} \quad (4.10)$$

In a similar fashion one can show that for $x \in [x_2, \infty)$ the position of the wave front is given by

$$\begin{aligned} W(t_*, x) = 0 &\Rightarrow ct_* - \inf_{0 \leq \mu_0 \leq \mu_1 \leq t_*} \left\{ \frac{A+B}{4} \frac{(x - x_2)^2}{\mu_0} + \frac{A}{4} \frac{x_2^2}{t_* - \mu_1} \right\} = 0 \\ &\Rightarrow t_*(x) = \sqrt{\frac{A}{4c}} \left[x_2 + \frac{\sqrt{A+B}}{\sqrt{A}} (x - x_2) \right]. \end{aligned} \quad (4.11)$$

We make the following remarks.

Remark 4.3. When $u(x)$ and $v(x)$ are smooth linear functions, say for example $u(x) = x$ and $v(x) = Ax$, then the $D_v D_u$ operator corresponds to a standard Wiener process with a diffusion coefficient that depends on the slopes of u and v . In particular, for the case $u(x) = x$ and $v(x) = Ax$, we have that $D_v D_u = \frac{1}{A} \frac{d^2}{dx^2}$, the diffusion coefficient is $\sqrt{\frac{2}{A}}$ and, as it is well known (see for example [14] and [5]), the front travels with constant K-P-P speed $\sqrt{\frac{4c}{A}}$. However, as we can see from equations (4.9)-(4.10) and (4.11), the corner points of u and v functions cause a change in the speed of propagation of the front. In particular, in the example considered above, the wave front travels with speed $\sqrt{\frac{4c}{A}}$ for $x < x_2$ and with speed $\sqrt{\frac{4c}{A+B}}$ for $x > x_2$. Namely, the speed of propagation is different for different areas of the semi-axis $\{x : x > 0\}$.

□

Remark 4.4. Moreover, a careful inspection of the calculations above shows the quite remarkable result that even though the function v has a discontinuity at the point $x = x_1$, the action functional, evaluated at the function ϕ that attains the supremum of (4.6), does not see this. This implies that the discontinuity of v at the point $x = x_1$ does not affect the propagation of the wave front and, in particular, it does not cause delay of the wave front. By delay of the wave front we mean the situation where the wave front stays on a particular point for a positive amount of time. At first sight, this is counterintuitive since one would expect the wave front to experience delay at this point because the underlying process has delay at $x = x_1$. However, as we saw, this is not true for this case. See the next section for some more detailed discussion on this.

□

Remark 4.5. One may also assume that $c(x)$ is not homogeneous in x . For example, one may suppose that $c(x) = c_1 > 0$ for $x < x^*$ and $c(x) = c_2 > 0$ for $x > x^*$, where $0 < c_1 < c_2$ are constants and x^* is some point on the positive x -axis. It is well known, [5], that in the case of standard reaction-diffusion equations, i.e. when the operator is the standard second order elliptic operator, the condition $c_2 > 2c_1$ leads to jumps of the wave front (for more details see [5]). It is easy to see that the aforementioned effects carry out in the case of generalized reaction diffusions as well.

□

Remark 4.6. In this example we assumed that $u(x) = x$ just for brevity. Of course, one could also assume that u has corner points. Then the phenomena that one observes are similar to the ones described above. Moreover, one can easily extend the aforementioned to the case where u and v have more than one non smoothness points.

□

These complete the study of wave front propagation for piecewise linear functions u and v .

5 Concluding remarks

In this paper we considered the large deviations principle for a large class of one dimensional strong Markov processes that are continuous with probability one. These processes were uniquely characterized by Feller [3] by a generalized second order differential operator $D_v D_u$ and its domain of definition. We derived the action functional for a strong Markov process X_t^ϵ with operator $\epsilon D_v D_u$. Of course, such a process can be derived by the process X_t that is governed by the operator $D_v D_u$ through a time change $t \rightarrow \epsilon t$, i.e. $X_t^\epsilon = X_{\epsilon t}$. We also considered reaction diffusion equations whose operator is a $D_v D_u$ operator and studied the problem of wave front propagation for K-P-P type generalized reaction diffusion equations in a simple but intuitive setting.

However, the following questions arise naturally.

- (i). The process that we considered is governed by an operator of the form $\epsilon D_v D_u$, i.e. the ϵ multiplies the operator and the functions v and u are independent of ϵ . A natural question arises. What kind of dependence of the functions v and u on ϵ would guarantee a large deviations principle for the resulting process ? Related to the latter question is also the following. How could one incorporate the drift in the action functional in this general setting ? In other words, what is the right formulation of the problem, which would include the usual case (1.4), with the drift term $b(\cdot)$ present, as a special case ?
- (ii). What other phenomena could one observe due to the non-smoothness points of u and v functions ? For example, in what scenario would the wave front have delay at particular points ? It is natural to expect delay at points of discontinuity of the function v , since at these points the corresponding process has delay. However, as we saw in the previous section, the simple situation where v has finitely many discontinuity points and it is independent of ϵ does not give delay of the front. The same is true even if we assume that v is discontinuous at every integer point for example. This is because the front has the “tendency” to propagate forward and this scenario is not sufficient to “slow down” the front at these points. One, probably, needs to consider a more involved situation where u and/or v functions would also depend on ϵ .

We plan to address these questions in a future work.

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